Computing isogenies and endomorphism rings of supersingular elliptic curves

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Joint work with Kirsten Eisenträger, Sean Hallgren, Kristin Lauter, Christophe Petit

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- Both protocols mentioned above use supersingular elliptic curves, but the problems considered in this talk pertain to SIDH, or the hash function of Charles-Goren-Lauter, rather than CSIDH

Isogenies

Let k be a finite field of characteristic $p > 3$, and let E, E' be two elliptic curves over k.

- An *isogeny over k* is a surjective morphism

$$\phi : E \to E',$$

  defined over k, which induces a group homomorphism from $E(\overline{k}) \to E'(\overline{k})$.
- Every finite subgroup $K \subseteq E(\overline{k})$ determines a separable isogeny $\phi : E \to E/K$, unique up to isomorphism

The endomorphism ring

- An *endomorphism* of E is an isogeny $\phi : E \to E$, possibly defined over an extension of k.
- Let $\mathrm{End}(E)$ $(= \mathrm{End}_{\overline{k}}(E))$ be the set of endomorphisms of E, together with the zero map on E.
- $\mathrm{End}(E)$ is a ring: addition is defined pointwise, and multiplication is given by composition.
- $\mathrm{End}(E)$ always contains $\mathbb{Z}$: let $n \in \mathbb{Z}$, then the multiplication-by-n map

$$\begin{aligned} [n] : E &\to E \\ P &\mapsto \underbrace{P + \cdots + P}_{n \text{ times}} \end{aligned}$$

  is an endomorphism of E.

Supersingular elliptic curves

Definition

E/k is *supersingular* if its endomorphism algebra

$$B := \mathrm{End}(E) \otimes \mathbb{Q}$$

is a quaternion algebra over $\mathbb{Q}$, i.e. a central simple $\mathbb{Q}$-algebra of dimension 4 over $\mathbb{Q}$.

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- The j-invariant of a supersingular elliptic curve defined over $\overline{\mathbb{F}_p}$ is in $\mathbb{F}_{p^2}$.
- There are $\lfloor \frac{p-1}{12} \rfloor + \epsilon$ supersingular j-invariants in $\mathbb{F}_{p^2}$, where $\epsilon \in \{0, 1, 2\}$.

SIDH and the CGL hash function

- A private key in SIDH or the CGL hash is an ℓ-power isogeny $\phi : E \to E'$ between two supersingular curves $E, E'/\mathbb{F}_{p^2}$, for distinct primes p, ℓ.

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- A private key in SIDH or the CGL hash is an ℓ-power isogeny $\phi : E \to E'$ between two supersingular curves $E, E'/\mathbb{F}_{p^2}$, for distinct primes p, ℓ.
- Computing such an isogeny amounts to path finding in supersingular isogeny graphs.

Supersingular isogeny graphs

Let $\Phi_\ell(X, Y)$ be the ℓth modular polynomial.

Definition

Let p, ℓ be distinct primes. The graph $G(p, \ell)$ has as its vertices supersingular j-invariants, and the number of edges from j to j' is the multiplicity of j' as a root of $\Phi_\ell(j, Y)$.

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- vertices are a complete set of representaives of the isomorphism classes of supersingular elliptic curves,
- the edges from E to E' are ℓ-isogenies $\phi : E \to E'$
- (we identify two isogenies ϕ_1, ϕ_2 if $\phi_1 = u \circ \phi_2$ for some $u \in \mathrm{Aut}(E')$.)

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Pathfinding in $G(p, \ell)$ is equivalent to computing an ℓ-power isogeny between two given supersingular elliptic curves.

The isogeny graph $G(157, 3)$

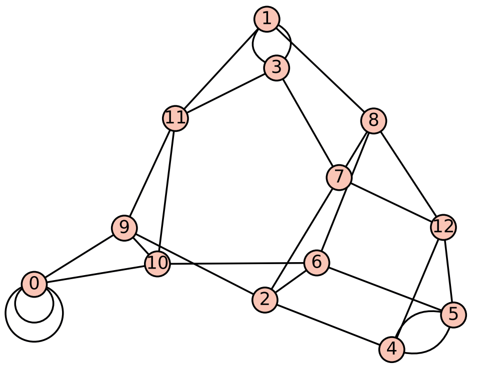

Figure: $G(157, 3)$

Pathfinding in $G(p, \ell)$ and computing endomorphisms

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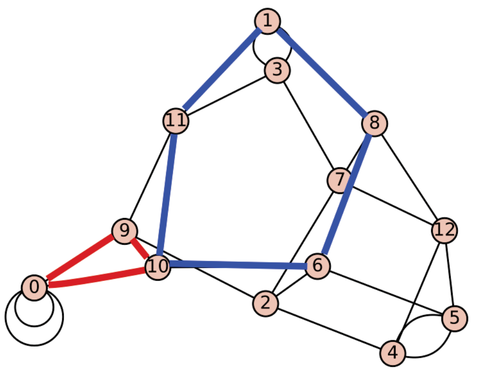

Figure: $\langle 1, \alpha, \beta, \alpha\beta \rangle = \Lambda \subseteq \mathrm{End}(E)$ is an order

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Theorem (Eisenträger, Hallgren, Lauter, M-, Petit)

Assume $\ell = O(\log p)$. Then there are polynomial-time (in $\log p$) reductions between the problem of pathfinding in $G(p,\ell)$ and computing endomorphism rings of supersingular elliptic curves, assuming some heuristics.

Quaternion algebras

- Every *quaternion algebra over* $\mathbb{Q}$ is of the form, for some $a, b \in \mathbb{Q}^\times$,

$$H(a,b) := \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$$

  where $i^2 = a, j^2 = b$, and $ij = -ji$.

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  where $i^2 = a, j^2 = b$, and $ij = -ji$.
- $H(a,b)$ has an *involution* sending

$$\alpha = w + xi + yj + zij \mapsto \overline{\alpha} := w - xi - yj - zij.$$

  This lets us define the *reduced norm* and *reduced trace* of an element α:

$$\begin{aligned} \mathrm{nrd}(\alpha) &:= \alpha\overline{\alpha} = w^2 - ax^2 - by^2 + abz^2 \\ \mathrm{trd}(\alpha) &:= \alpha + \overline{\alpha} = 2w. \end{aligned}$$

Let $B/\mathbb{Q}$ be a quaternion algebra and let v be a place of $\mathbb{Q}$.
Let H_v be the 4-dimensional division algebra over $\mathbb{Q}_v$.

$$B \otimes \mathbb{Q}_v \simeq \begin{cases} M_2(\mathbb{Q}_v) & \text{we say } B \text{ is } \textit{split} \text{ at } v \\ H_v & \text{we say } B \text{ is } \textit{ramified} \text{ at } v. \end{cases}$$

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For example:

- $H(-1,-1)$ is ramified at $\{2,\infty\}$.
- Let $p \equiv 3 \pmod 4$ be a prime. Then $H(-1,-p)$ is ramified at $\{p,\infty\}$.

The endomorphism algebra

Again, let k be a finite field, $\mathrm{char}(K) = p > 3$.

- Assume E/k is supersingular. Then $\mathrm{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra ramified exactly at $\{p,\infty\}$, and the standard involution is given by taking duals, so $\mathrm{nrd} = \deg$.

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- We can say more: $\mathrm{End}(E)$ is a maximal order in $\mathrm{End}(E) \otimes \mathbb{Q}$.
- If E/k is ordinary, $\mathrm{End}(E)$ is a quadratic (but not necessarily maximal) order in its endomorphism algebra, a quadratic imaginary extension of $\mathbb{Q}$

An example

Let $p \equiv 3 \pmod 4$ be a prime. Let $E/\mathbb{F}_p$ be the elliptic curve $E : y^2 = x^3 + x$. We have the endomorphisms

$$\phi : (x, y) \mapsto (-x, \sqrt{-1}y)$$
$$\pi : (x, y) \mapsto (x^p, y^p).$$

- The map $\phi \mapsto i$, $\pi \mapsto j$ extends linearly to an isomorphism of quaternion algebras $\mathrm{End}(E) \otimes \mathbb{Q} \simeq H(-1, -p)$.
- However: $\langle 1, \phi, \pi, \phi\pi \rangle \subsetneq \mathrm{End}(E)$.

Arithmetic of endomorphism rings and isogenies

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- Suppose that $\phi : E \to E'$ is an isogeny. Then

$$\iota : \mathrm{End}(E') \hookrightarrow \mathrm{End}(E) \otimes \mathbb{Q}$$
$$\rho \mapsto \left(\widehat{\phi} \circ \rho \circ \phi\right) \otimes \frac{1}{\deg \phi}$$

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- Set $I := \{\alpha \in \mathrm{End}(E) : \alpha(\ker \phi) = \{0\}\}$. This is a left ideal of $\mathrm{End}(E)$, and $\deg(\phi) = \mathrm{nrd}(I)$.

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- Then $\mathrm{End}(E')$ is isomorphic to the *right order of I*:

$$\mathcal{O}_R(I) := \{\gamma \in \mathrm{End}(E) \otimes \mathbb{Q} : I\gamma \subseteq I\} = \iota(\mathrm{End}(E'))$$

Arithmetic of endomorphism rings and isogenies

- Conversely, given a left ideal $I \subseteq \mathrm{End}(E)$ such that $\mathrm{nrd}(I)$ is coprime to p, define

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- We have $\mathrm{nrd}(I) = |E[I]| = \deg(\phi_I)$.

Computing ℓ-power isogenies

Problem

Given distinct primes p, ℓ and supersingular elliptic curves $E/\mathbb{F}_{p^2}$ and $E'/\mathbb{F}_{p^2}$, compute an isogeny $\phi : E \to E'$ whose degree is ℓ^e for some e.

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- This problem can return an isogeny of size polynomial in $\log p$ if $\ell = O(\log p)$: we can represent ϕ by a sequence of ℓ-isogenies, and the diameter of $G(p, \ell)$ is $O(\log p)$.
- This is the problem of pathfinding in $G(p, \ell)$.

Computing endomorphism rings

We can interpret the problem of "computing the endomorphism ring" in different ways: for example, we could ask for the geometric object $\mathrm{End}(E)$. We will simply ask for an order in $B_{p,\infty}$ isomorphic to $\mathrm{End}(E)$. Here $B_{p,\infty}$ denotes the quaternion algebra ramified at $\{p, \infty\}$.

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For a polynomial-time reduction from computing isogenies to this problem to make sense, we need to know that such an order $\mathcal{O}$ of polynomial size exists.

Endomorphism rings have polynomial size

Theorem (Eisenträger, Hallgren, Lauter, M-, Petit)

Every isomorphism class (i.e. conjugacy class) of maximal orders in $B_{p,\infty}$ contains an order $\mathcal{O}$ of size polynomial in $\log p$.

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- The map $[I] \mapsto [\mathcal{O}_R(I)]$ from left ideal classes of $\mathcal{O}$ to isomorphism classes of maximal orders is surjective
- Every left ideal class contains a representative J such that $\mathrm{nrd}(J) = O(p^2)$

Almost equivalent problems, categorically

Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$.

Problem

Let $\mathcal{O}, \mathcal{O}' \subseteq B_{p,\infty}$ be maximal orders. Let $\ell \neq p$ be a prime. Compute a left ideal $I \subseteq \mathcal{O}$ such that $\mathcal{O}_R(I) \simeq \mathcal{O}'$.

- If $\mathcal{O}, \mathcal{O}'$ have size polynomial in $\log p$, and $\ell = O(\log p)$, then an algorithm of Kohel-Lauter-Petit-Tignol solves this problem in time polynomial in $\log p$
- Why almost? If $E/\overline{\mathbb{F}_p}, E'/\overline{\mathbb{F}_p}$ are supersingular, then $\mathrm{End}(E) \simeq \mathrm{End}(E')$ if and only if $j(E)^p = j(E')$.

Computing isogenies reduces to computing endomorphism rings

Assume we have an oracle which, on input $E/\mathbb{F}_{p^2}$ supersingular, computes a maximal order $\mathcal{O} \subset B_{p,\infty}$ such that $\mathcal{O} \simeq \mathrm{End}(E)$. Suppose we are given two supersingular elliptic curves $E, E'/\mathbb{F}_{p^2}$ and a prime $\ell = O(\log p)$. We sketch an algorithm for computing an ℓ-power isogeny $\phi : E \to E'$.

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Now we want to translate the orders $\mathcal{O}_k$ into a sequence of ℓ-isogenies.

Translating $\mathcal{O}_1, \ldots, \mathcal{O}_e$ to isogenies

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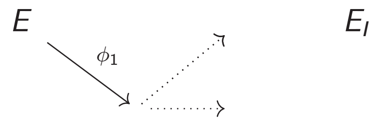

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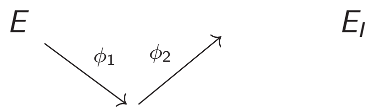

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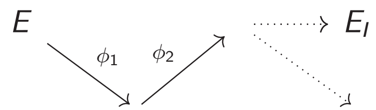

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- Return the sequence of isogenies $\phi_1, \dots, \phi_e$.

One issue: let $\phi_I : E \to E_I$ be the isogeny corresponding to the path in $G(p, \ell)$ constructed in the reduction. We have $\mathrm{End}(E_I) \simeq \mathrm{End}(E')$, but it could be that $E_I \simeq (E')^{(p)}$ (i.e. $j(E_I)^p = j(E') \neq j(E_I)$).

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- Compute an ideal of ℓ-power norm equivalent to IP and repeat the algorithm.

Thank you!