Computing isogenies and endomorphism rings of supersingular elliptic curves

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Joint work with Kirsten Eisenträger, Sean Hallgren, Kristin Lauter, Christophe Petit

Elliptic curves and post-quantum cryptography

- A quantum computer could efficiently calculate discrete logs of points on elliptic curves
- Elliptic curve cryptography is insecure in a "post-quantum" world

There are several proposed isogeny based public key cryptosystems which could remain secure. For example, we are learning about SIDH and CSIDH at this conference.
Elliptic curves and post-quantum cryptography

- A quantum computer could efficiently calculate discrete logs of points on elliptic curves
- Elliptic curve cryptography is insecure in a “post-quantum” world
- There are several proposed isogeny based public key cryptosystems which could remain secure. For example, we are learning about SIDH and CSIDH at this conference
- Secret keys are isogenies between elliptic curves defined over finite fields

Isogenies

Let $k$ be a finite field of characteristic $p > 3$, and let $E, E'$ be two elliptic curves over $k$.

- An isogeny over $k$ is a surjective morphism
  $\phi : E \to E'$,

  defined over $k$, which induces a group homomorphism from $E(\overline{k}) \to E'(\overline{k})$.

- Every finite subgroup $K \subseteq E(\overline{k})$ determines a separable isogeny $\phi : E \to E/K$, unique up to isomorphism

The endomorphism ring

- An endomorphism of $E$ is an isogeny $\phi : E \to E$, possibly defined over an extension of $k$.
- Let $\text{End}(E) (= \text{End}_k(E))$ be the set of endomorphisms of $E$, together with the zero map on $E$.
- $\text{End}(E)$ is a ring: addition is defined pointwise, and multiplication is given by composition.
- $\text{End}(E)$ always contains $\mathbb{Z}$: let $n \in \mathbb{Z}$, then the multiplication-by-$n$ map

$$ [n] : E \to E \\
P \mapsto P + \cdots + P \quad (n \text{ times}) $$

is an endomorphism of $E$. 
Supersingular elliptic curves

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- The $j$-invariant of a supersingular elliptic curve defined over $\mathbb{F}_p$ is in $\mathbb{F}_{p^2}$.
- There are $\lfloor \frac{p-1}{12} \rfloor + \epsilon$ supersingular $j$-invariants in $\mathbb{F}_{p^2}$, where $\epsilon \in \{0, 1, 2\}$.

SIDH and the CGL hash function

- A private key in SIDH or the CGL hash is an $\ell$-power isogeny $\phi : E \rightarrow E'$ between two supersingular curves $E, E'/\mathbb{F}_{p^2}$, for distinct primes $p, \ell$.
- Computing such an isogeny amounts to path finding in supersingular isogeny graphs.
Supersingular isogeny graphs

Let $\Phi_\ell(X, Y)$ be the $\ell$th modular polynomial.

**Definition**

Let $p, \ell$ be distinct primes. The graph $G(p, \ell)$ has as its vertices supersingular $j$-invariants, and the number of edges from $j$ to $j'$ is the multiplicity of $j'$ as a root of $\Phi_\ell(j, Y)$.

Another way to think about $G(p, \ell)$:
- vertices are a complete set of representatives of the isomorphism classes of supersingular elliptic curves,
- the edges from $E$ to $E'$ are $\ell$-isogenies $\phi: E \to E'$
- (we identify two isogenies $\phi_1, \phi_2$ if $\phi_1 = u \circ \phi_2$ for some $u \in \text{Aut}(E')$.)
Properties of $G(p, \ell)$

- $G(p, \ell)$ has $O(p)$ vertices, and every vertex has out-degree $\ell + 1$
- $G(p, \ell)$ is connected and its diameter is $O(\log p)$
- If $p \equiv 1 \pmod{12}$, the graph is an undirected $(\ell + 1)$-regular Ramanujan graph

Pathfinding in $G(p, \ell)$ is equivalent to computing an $\ell$-power isogeny between two given supersingular elliptic curves.
The isogeny graph $G(157, 3)$

Pathfinding in $G(p, \ell)$ and computing endomorphisms

Kohel gave an algorithm which, given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$, computes an order $\Lambda \subseteq \text{End}(E)$.

Computing isogenies and endomorphism rings

- Pathfinding in $G(p, \ell)$ lets one compute endomorphisms of supersingular elliptic curves.

Figure: $\langle 1, \alpha, \beta, \alpha \beta \rangle = \Lambda \subseteq \text{End}(E)$ is an order
Computing isogenies and endomorphism rings

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- Conversely, pathfinding in $G(p, \ell)$ reduces to the problem of computing endomorphism rings.

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- Conversely, pathfinding in $G(p, \ell)$ reduces to the problem of computing endomorphism rings.

**Theorem (Eisenträger, Hallgren, Lauter, M-, Petit)**

Assume $\ell = O(\log p)$. Then there are polynomial-time (in $\log p$) reductions between the problem of pathfinding in $G(p, \ell)$ and computing endomorphism rings of supersingular elliptic curves, assuming some heuristics.

Quaternion algebras

- Every quaternion algebra over $\mathbb{Q}$ is of the form, for some $a, b \in \mathbb{Q}^\times$,

$$H(a, b) := \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$$

where $i^2 = a, j^2 = b$, and $ij = -ji$.

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where $i^2 = a, j^2 = b$, and $ij = -ji$.
- $H(a, b)$ has an involution sending

$$\alpha = w + xi + yj + ziz \mapsto \overline{\alpha} := w - xi - yj - zij.$$

This lets us define the reduced norm and reduced trace of an element $\alpha$:

$$\text{nr}(\alpha) := \alpha \overline{\alpha} = w^2 - ax^2 - by^2 + abz^2$$
$$\text{tr}(\alpha) := \alpha + \overline{\alpha} = 2w.$$
Let $B/\mathbb{Q}$ be a quaternion algebra and let $v$ be a place of $\mathbb{Q}$. Let $H_v$ be the 4-dimensional division algebra over $\mathbb{Q}_v$. 

\[ B \otimes \mathbb{Q}_v \cong \begin{cases} 
M_2(\mathbb{Q}_v) & \text{we say } B \text{ is split at } v \\
H_v & \text{we say } B \text{ is ramified at } v.
\end{cases} \]

For example:

- $H(-1, -1)$ is ramified at $\{2, \infty\}$.
- Let $p \equiv 3 \pmod{4}$ be a prime. Then $H(-1, -p)$ is ramified at $\{p, \infty\}$.

The endomorphism algebra

Again, let $k$ be a finite field, $\text{char}(K) = p > 3$.

- Assume $E/k$ is supersingular. Then $\text{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra ramified exactly at $\{p, \infty\}$, and the standard involution is given by taking duals, so $\text{nr}d = \text{deg}$.

We can say more: $\text{End}(E)$ is a maximal order in $\text{End}(E) \otimes \mathbb{Q}$. 
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- Assume $E/k$ is supersingular. Then $\text{End}(E) \otimes \mathbb{Q}$ is a quaternion algebra ramified exactly at $\{p, \infty\}$, and the standard involution is given by taking duals, so $\text{nd} = \text{deg}$.
- We can say more: $\text{End}(E)$ is a maximal order in $\text{End}(E) \otimes \mathbb{Q}$.
- If $E/k$ is ordinary, $\text{End}(E)$ is a quadratic (but not necessarily maximal) order in its endomorphism algebra, a quadratic imaginary extension of $\mathbb{Q}$.

An example

Let $p \equiv 3 \pmod{4}$ be a prime. Let $E/\mathbb{F}_p$ be the elliptic curve $E : y^2 = x^3 + x$. We have the endomorphisms

\[
\begin{align*}
\phi : (x, y) &\mapsto (-x, \sqrt{-1}y) \\
\pi : (x, y) &\mapsto (x^p, y^p).
\end{align*}
\]

- The map $\phi \mapsto i$, $\pi \mapsto j$ extends linearly to an isomorphism of quaternion algebras $\text{End}(E) \otimes \mathbb{Q} \cong H(-1, -p)$.
- However: $\langle 1, \phi, \pi, \phi \pi \rangle \subsetneq \text{End}(E)$.

Arithmetic of endomorphism rings and isogenies

Work of Waterhouse connects the arithmetic of $\text{End}(E)$ to isogenies $\phi : E \to E'$. Let $E/\mathbb{F}_{p^2}$ be supersingular.

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- Suppose that $\phi : E \to E'$ is an isogeny. Then

\[
\begin{align*}
\iota : \text{End}(E') &\hookrightarrow \text{End}(E) \otimes \mathbb{Q} \\
\rho &\mapsto \left( \hat{\phi} \circ \rho \circ \phi \right) \otimes \frac{1}{\text{deg} \phi}
\end{align*}
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embeds $\text{End}(E')$ as a maximal order in $\text{End}(E) \otimes \mathbb{Q}$. 
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embeds $\text{End}(E')$ as a maximal order in $\text{End}(E) \otimes \mathbb{Q}$.

- Set $I := \{ \alpha \in \text{End}(E) : \alpha(\ker \phi) = \{0\} \}$. This is a left ideal of $\text{End}(E)$, and $\deg(\phi) = \text{nrd}(I)$.

Arithmetic of endomorphism rings and isogenies

- Conversely, given a left ideal $I \subseteq \text{End}(E)$ such that $\text{nrd}(I)$ is coprime to $\rho$, define

$$E[I] := \bigcap_{\alpha \in I} \ker \alpha.$$  

$$E[I]$$ is a finite subgroup of $E(\mathbb{F}_p^2)$ and thus determines an isogeny

$$\phi_I : E \to E_I := E/E[I].$$
Arithmetic of endomorphism rings and isogenies

- Conversely, given a left ideal $I \subseteq \text{End}(E)$ such that $\text{nrd}(I)$ is coprime to $p$, define
  $$E[I] := \bigcap_{\alpha \in I} \ker \alpha.$$  

- $E[I]$ is a finite subgroup of $E(\mathbb{F}_{p^2})$ and thus determines an isogeny
  $$\phi_I : E \to E_I := E/E[I].$$  

- We have $\text{nrd}(I) = |E[I]| = \deg(\phi_I)$.

Computing $\ell$-power isogenies

Problem

Given distinct primes $p$, $\ell$ and supersingular elliptic curves $E/\mathbb{F}_{p^2}$ and $E'/\mathbb{F}_{p^2}$, compute an isogeny $\phi : E \to E'$ whose degree is $\ell^e$ for some $e$.

- This problem can return an isogeny of size polynomial in $\log p$ if $\ell = O(\log p)$: we can represent $\phi$ by a sequence of $\ell$-isogenies, and the diameter of $G(p, \ell)$ is $O(\log p)$.
- This is the problem of pathfinding in $G(p, \ell)$.

Computing endomorphism rings

We can interpret the problem of "computing the endomorphism ring" in different ways: for example, we could ask for the geometric object $\text{End}(E)$. We will simply ask for an order in $B_{p,\infty}$ isomorphic to $\text{End}(E)$. Here $B_{p,\infty}$ denotes the quaternion algebra ramified at $\{p, \infty\}$. 
Computing endomorphism rings

We can interpret the problem of “computing the endomorphism ring” in different ways: for example, we could ask for the geometric object End(E). We will simply ask for an order in \( B_{p,\infty} \) isomorphic to End(E). Here \( B_{p,\infty} \) denotes the quaternion algebra ramified at \( \{p, \infty\} \).

\[ \text{Problem} \]

Given a supersingular elliptic curve \( E/\mathbb{F}_p^2 \), compute an order \( O \subseteq B_{p,\infty} \) such that \( \text{End}(E) \simeq O \).

For a polynomial-time reduction from computing isogenies to this problem to make sense, we need to know that such an order \( O \) of polynomial size exists.

Endomorphism rings have polynomial size

Theorem (Eisenträger, Hallgren, Lauter, M-, Petit)

Every isomorphism class (i.e. conjugacy class) of maximal orders in \( B_{p,\infty} \) contains an order \( O \) of size polynomial in \( \log p \).

Sketch of proof:
- Pizer shows \( B_{p,\infty} \) and at least one maximal order \( O_0 \subseteq B_{p,\infty} \) have polynomial in \( \log p \) size.
Endomorphism rings have polynomial size

Theorem (Eisentraeger, Hallgren, Lauter, M-, Petit)

Every isomorphism class (i.e. conjugacy class) of maximal orders in $B_{p,\infty}$ contains an order $\mathcal{O}$ of size polynomial in $\log p$.

Sketch of proof:
- Pizer shows $B_{p,\infty}$ and at least one maximal order $\mathcal{O}_0 \subseteq B_{p,\infty}$ have polynomial in $\log p$ size
- The map $[I] \mapsto [O_R(I)]$ from left ideal classes of $\mathcal{O}$ to isomorphism classes of maximal orders is surjective

Almost equivalent problems, categorically

Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ ramified at $\{p, \infty\}$.

Problem

Let $\mathcal{O}, \mathcal{O}' \subseteq B_{p,\infty}$ be maximal orders. Let $\ell \neq p$ be a prime. Compute a left ideal $I \subseteq \mathcal{O}$ such that $O_R(I) \simeq \mathcal{O}'$.

- If $\mathcal{O}, \mathcal{O}'$ have size polynomial in $\log p$, and $\ell = O(\log p)$, then an algorithm of Kohel-Lauter-Petit-Tignol solves this problem in time polynomial in $\log p$
- Why almost? If $E/\mathbb{F}_p, E'/\mathbb{F}_p$ are supersingular, then $\text{End}(E) \simeq \text{End}(E')$ if and only if $j(E)^\ell = j(E')$.

Computing isogenies reduces to computing endomorphism rings

Assume we have an oracle which, on input $E/\mathbb{F}_p^2$ supersingular, computes a maximal order $\mathcal{O} \subset B_{p,\infty}$ such that $\mathcal{O} \simeq \text{End}(E)$. Suppose we are given two supersingular elliptic curves $E, E'/\mathbb{F}_p^2$ and a prime $\ell = O(\log p)$. We sketch an algorithm for computing an $\ell$-power isogeny $\phi : E \to E'$. 
Computing isogenies reduces to computing endomorphism rings

1. Compute \( \mathcal{O} \simeq \text{End}(E), \mathcal{O}' \simeq \text{End}(E') \)

2. Compute a left ideal \( I \subseteq \mathcal{O} \) such that \( \mathcal{O}_R(I) \simeq \mathcal{O}' \), \( \text{nrd}(I) = \ell^e \) using KLPT

3. Compute the ideals \( I_k := I + \ell^k \mathcal{O} \); \( \text{nrd}(I_k) = \ell^k \).

4. Compute the orders \( \mathcal{O}_k := \mathcal{O}_R(I_k) \)
Computing isogenies reduces to computing endomorphism rings

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Now we want to translate the orders \( \mathcal{O}_k \) into a sequence of \( \ell \)-isogenies.

Translating \( \mathcal{O}_1, \ldots, \mathcal{O}_e \) to isogenies

\[
\begin{array}{ccc}
E & \longrightarrow & E_i \\
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Translating $O_1, \ldots, O_e$ to isogenies

\[ E \xrightarrow{\phi_1} E_1 \xrightarrow{\phi_2} E_2 \xrightarrow{\phi_3} E_3 \]

- At step $k$, we compute the neighbors
- Then we check which neighbor’s endomorphism ring is isomorphic to $O_R(l_k)$

One issue: let $\phi_l : E \to E_l$ be the isogeny corresponding to the path in $G(p, \ell)$ constructed in the reduction. We have $\text{End}(E_l) \simeq \text{End}(E')$, but it could be that $E_l \simeq (E')^p (i.e. j(E_l)^p = j(E')^p \neq j(E_l))$.

In this case, we replace $l$ with $l \cdot P$, where $P \subseteq O_R(l)$ is the unique 2-sided ideal of norm $p$.
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- In this case, we replace $I$ with $I \cdot P$, where $P \subseteq \mathcal{O}_R(I)$ is the unique 2-sided ideal of norm $p$.
- Compute an ideal of $\ell$-power norm equivalent to $IP$ and repeat the algorithm.

Thank you!